ON THE REAL SOLUTIONS OF SYSTEMS OF TWO HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER*

BY

MAXIME BÔCHER

The close relation which exists between the system of two homogeneous linear differential equations of the first order:

$$(1) \qquad \begin{aligned} y' &= Py - Qz, \\ z' &= Ry - Sz, \end{aligned}$$

and a single homogeneous linear differential equation of the second order:

$$(2) \qquad y'' + py' + qy = 0,$$

is well known. Thus, if we let $z = y'$, we get as a system of equations equivalent to (2):

$$(3) \qquad \begin{aligned} y' &= z, \\ z' &= -qy - pz, \end{aligned}$$

and this is merely a special case of (1). On the other hand, if we eliminate z from (1), we obtain the equation:

$$(4) \qquad y'' + \left(S - P - \frac{Q'}{Q}\right)y' + \left(QR - PS - P' + \frac{PQ'}{Q}\right)y = 0,$$

which comes under the form (2).

The system (1) and the single equation (2) are not, however, for this reason in all cases equivalent to each other. It is true that any theorem concerning (1) when applied to (3) yields a theorem concerning (2). Conversely, however, we cannot always obtain from a theorem concerning (2), by applying it to (4), a theorem concerning (1); since in the first place (4) has no meaning unless P and Q have first derivatives; and in the second place, unless we are willing to consider the case in which the coefficients p and q of (2) are discontinuous, we must impose on P and Q the further conditions that P' and Q' be continuous and that Q do not vanish—restrictions which, if we treat (1) directly, are quite unnecessary.

* Presented to the Society December 28, 1901. Received for publication February 7, 1902.

It appears therefore that the system (1) is more general than equation (2).

It is the object of the present paper to establish a series of propositions concerning the system (1) analogous to the theorems concerning (2) which were first given by STURM in the first volume of Liouville's Journal (1836). STURM's original theorems, in some cases considerably generalized, follow as mere corollaries from the theorems we shall obtain, as will be indicated briefly in § 7. Moreover the method here used has the advantage not only of generality but also of simplicity—an advantage which is particularly apparent when we take the standpoint of modern rigor.

§ 1. *Recapitulation of certain known theorems. Terminology and notation.*

We shall be concerned exclusively with the case in which the independent variable x is real and is confined to the interval:

$$(I) \qquad a \leqq x \leqq b.$$

Throughout (I) the coefficients P, Q, R, S of (1) are assumed to be real and continuous, but not necessarily analytic, functions of x.

The fundamental existence theorem for (1) is the following:*

If x_0 is any point of (I) there exists one and only one pair of functions (y, z) continuous, having continuous first derivatives, and satisfying (1) *at every point of (I), and having at x_0 the arbitrarily given values (y_0, z_0).*

A special case of this theorem which we shall have frequent occasion to use is the following: †

If (y, z) is a solution of (1), *and if y and z both vanish at a single point of (I), they both vanish identically throughout (I).*

From this fol ows:

If $y \equiv 0$ then either $Q \equiv 0$ or $z \equiv 0$.

For if z vanishes at any point of (I) it must, by the last theorem, vanish identically, and if z vanishes nowhere in (I) it follows from the first equation (1) that $Q \equiv 0$. Similarly:

If $z \equiv 0$ then either $R \equiv 0$ or $y \equiv 0$.

The two theorems ust stated illustrate a duality which exists owing to the symmetry of equations (1). Whenever a theorem concerning y has been obtained it is possible, by making slight and obvious changes, to obtain a theorem concerning z. We shall in future confine ourselves to the theorems concerning y.

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*PEANO, Mathematische Annalen, vol. 32 (1888), p. 450.

†This theorem can readily be deduced from formula (5) below by a method similar to that used by STURM, l. c., pp. 109–110.

Let (y_1, z_1) and (y_2, z_2) be two solutions of (1) and write:

$$D = y_1 z_2 - y_2 z_1 .$$

It is readily found that this determinant satisfies the differential equation:

$$D' = (P - S) D .$$

Accordingly: *

(5) $$D = k e^{\int (P-S)\,dx} .$$

This formula shows that D either vanishes nowhere or vanishes identically. In the former case (y_1, z_1) and (y_2, z_2) are linearly independent, in the latter linearly dependent.

We shall in the course of our work frequently have occasion to consider continuous real functions $F(x)$ which do not change sign in (I) *and do not vanish at all points of a connected portion of* (I). This state of affairs will be indicated by saying that $F(x)$ has a *characteristic sign* in (I); this sign being *plus* if $F(x) \geqq 0$, *minus* if $F(x) \leqq 0$. Using this terminology we may state the following lemma which is fundamental for much of our work.

LEMMA I: *If throughout* (I) p *and* r *are real and continuous functions of* x, *and if* r *has a characteristic sign, no solution of the differential equation:*

(6) $$y' = py + r ,$$

vanishes more than once in (I), *and if a solution vanishes at a point of* (I) *it increases or decreases with* x *at this point according as the characteristic sign of* r *is positive or negative.*

The proof of this lemma follows at once † by reference to the explicit formula for the solution of (6) by quadratures. In the same way we get also

LEMMA II: *If throughout* (I) p *and* r *are real and continuous functions of* x, *and if:*

$$r \left\{ \begin{matrix} \geqq \\ \leqq \end{matrix} \right\} 0 ,$$

then no solution of (6) *can pass with increasing* x *from* $\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}$ *values to zero, or from zero to* $\begin{matrix}\textit{negative}\\ \textit{positive}\end{matrix}$ *values.*

§ 2. *Concerning the roots of y and z.*

The fundamental theorem is the following in which, as well as in the subsequent theorems, (y, z) is supposed to be a solution of (1):

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* Formula (5) is the analogue of ABEL's formula for equation (2).

† Cf. vol. 2 of these Transactions (1901), pp. 434–435.

I. *If Q has a characteristic sign, then in a connected portion of (I) in which z does not vanish y cannot vanish more than once, and where it vanishes it changes sign.**

This theorem follows at once from Lemma I.

An immediate consequence of the last theorem is this:

II. *If Q has a characteristic sign,† and if y does not vanish identically, then y cannot vanish at an infinite number of points in (I).*

For if y has an infinite number of roots in (I) these roots have at least one limiting point c in (I), and owing to the continuity of y we must have $y(c) = 0$. But since y does not vanish identically $z(c) \neq 0$. Accordingly, since z is continuous, there is a certain neighborhood of c throughout which it does not vanish, and therefore, by the theorem last proved, y can vanish in this neighborhood only at c itself. This, however, involves a contradiction since by hypothesis c was a limiting point of roots of y.

By interchanging y and z in Theorem I we get a theorem concerning the case in which R has a characteristic sign and by combining these two results we get the proposition:

III. *If Q and R both have characteristic signs the roots of y and z separate each other, unless y and z are both identically zero.*

This last theorem however finds application only when the characteristic signs of Q and R are alike, as the following proposition shows:

IV. *If Q and R have opposite characteristic signs, and if y and z do not both vanish identically, then neither y nor z vanishes more than once in (I) and if one vanishes the other does not.*

For let:

$$w = yz.$$

This function satisfies the differential equation:

$$w' = (P - S)w + (Ry^2 - Qz^2). \tag{7}$$

Now if Q and R have opposite characteristic signs, and y and z do not vanish identically, the function $Ry^2 - Qz^2$ also has a characteristic sign, and we see, by applying Lemma I to the differential equation just written, that w cannot vanish more than once in (I). This proves our theorem.

Consider now two linearly independent solutions (y_1, z_1) and (y_2, z_2) of (1). We shall then prove:

V. *If Q has a characteristic sign the roots of y_1 and y_2 separate each other.*

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* This last clause does not of course apply to the ends a and b of (I).

† Or more generally: If (I) can be divided into a finite number of pieces in each of which Q has a characteristic sign.

In the first place it is clear that no root of y_1 can coincide with a root of y_2 as otherwise the determinant D of (5) would vanish, and this is impossible since (y_1, z_1) and (y_2, z_2) are linearly independent.

Next let us show that between two successive roots x_1 and x_2 of y_1 lies at least one root of y_2. If this were not the case the function:

$$f = \frac{y_1}{y_2},$$

would be continuous throughout the interval $x_1 \leqq x \leqq x_2$ and would vanish at the extremities of this interval but not otherwise in this interval. Accordingly the derivative of f must change sign in this interval.* But:

$$f' = \frac{y_1' y_2 - y_1 y_2'}{y_2^2} = \frac{QD}{y_2^2},$$

and therefore a change of sign of f' is impossible.

In the same way we see that between two successive roots of y_2 lies at least one root y_1. Thus our theorem is proved.

In conclusion consider the function:

$$\omega = \frac{z}{y},$$

of which constant use will be made in the next section. By multiplying the first equation (1) by z the second by y and subtracting we see that:

$$\text{(8)} \qquad \omega' = R - (P + S)\omega + Q\omega^2.$$

This is the Riccati's equation satisfied by ω at all points where ω is defined, i. e., where y is not zero.

As a simple application of (8) let us consider the case in which $P + S \equiv 0$ and Q and R have the same characteristic sign. Here we see from (8) that ω' also has the characteristic sign of Q and R, and, since in this case y cannot vanish an infinite number of times, we have the theorem:

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* We make use here of the following proposition which may be regarded as a supplement to ROLLE's theorem.

If the function $f(x)$ is real and continuous and has a finite derivative throughout the interval $x_1 \leqq x \leqq x_2$ and if $f(x_1) = f(x_2) = 0$ while $f(x)$ is not identically zero, then $f'(x)$ changes sign in the interval $x_1 < x < x_2$; i. e., there exist two points ξ_1 and ξ_2 in this interval such that $f'(\xi_1)$ and $f'(\xi_2)$ have opposite signs.

For let c be so chosen that $x_1 < c < x_2$ and $f(c) \neq 0$. Then by the law of the mean:

$$f(c) = (c - x_1) f'(\xi_1) = (c - x_2) f'(\xi_2) \qquad (x_1 < \xi_1 < c;\ c < \xi_2 < x_2).$$

Since $c - x_1$ and $c - x_2$ have opposite signs it follows from this equality that $f'(\xi_1)$ and $f'(\xi_2)$ have opposite signs.

VI. *If $P + S \equiv 0$ and Q and R have the same characteristic sign, then, except at the points (necessarily finite in number) where it becomes infinite, ω $\begin{smallmatrix}\text{increases}\\ \text{decreases}\end{smallmatrix}$ everywhere as x increases if the characteristic sign of Q and R is $\begin{smallmatrix}\text{positive}\\ \text{negative}\end{smallmatrix}$.*

§ 3. *Theorems of comparison.*

Consider now two pairs of equations:

$$(1_1)\qquad \begin{aligned} y' &= P_1 y - Q_1 z, \\ z' &= R_1 y - S_1 z, \end{aligned} \qquad\qquad (1_2)\qquad \begin{aligned} y' &= P_2 y - Q_2 z, \\ z' &= R_2 y - S_2 z, \end{aligned}$$

whose coefficients, which we assume to be continuous throughout (I), satisfy throughout (I) the relations:

$$(9)\qquad P_1 + S_1 = P_2 + S_2, \qquad Q_2 \geqq Q_1, \qquad R_2 \geqq R_1;$$

and let (y_1, z_1) and (y_2, z_2) be solutions of (1_1) and (1_2) respectively.

Let us first suppose that neither y_1 nor y_2 vanishes in (I) so that the functions:

$$\omega_1 = \frac{z_1}{y_1}, \qquad \omega_2 = \frac{z_2}{y_2}$$

are continuous throughout (I).

By subtracting the Riccati's equation satisfied by ω_1 from that satisfied by ω_2 it is seen that the function:

$$\eta = \omega_2 - \omega_1$$

satisfies the differential equation:

$$\eta' = -(P_1 + S_1)\eta + R_2 - R_1 + Q_2\omega_2^2 - Q_1\omega_1^2,$$

or, after a slight change of form:

$$(10)\qquad \eta' = -\left[P_1 + S_1 - \tfrac{1}{2}(Q_2 + Q_1)(\omega_2 + \omega_1)\right]\eta + \Delta,$$

where:

$$(11)\qquad \Delta = R_2 - R_1 + \tfrac{1}{2}(Q_2 - Q_1)(\omega_2^2 + \omega_1^2).$$

Since $\Delta \geqq 0$ we obtain the following theorem by applying to (10) the second lemma of § 1:

VII. *If the conditions* (9) *are fulfilled then the following three conditions:*

$$(12)\qquad \textit{Neither } y_1 \textit{ nor } y_2 \textit{ vanishes in } (I),$$

$$(13)\qquad \frac{z_2(a)}{y_2(a)} \geqq \frac{z_1(a)}{y_1(a)},$$

$$(14)\qquad \frac{z_2(b)}{y_2(b)} \leqq \frac{z_1(b)}{y_1(b)},$$

cannot all be satisfied except in the special case in which which $z_1/y_1 \equiv z_2/y_2$, *a case which can occur only when:*

$$R_1 \equiv R_2 \tag{15}$$

and

$$R_1 = R_2 = 0 \text{ and } z_1 = z_2 = 0, \text{ at all points where } Q_1 \neq Q_2.^* \tag{16}$$

This fundamental theorem admits, as will now be shown, an extension to the case in which y_1 or y_2 or both vanish at one or both ends of the interval (I) but at no other point of this interval. In this case ω_1 and ω_2 are continuous throughout (I) except that they may become positively or negatively infinite at the ends of this interval.

In order that the inequalities (13) and (14) should not become meaningless in the cases now to be considered we make the following conventions:

$$(17) \quad \begin{cases} (a) \text{ We write } \phi_2(c) > \phi_1(c) \\ \quad (1) \text{ when } \phi_2(c) = +\infty, \text{ and } \phi_1(c) = -\infty \text{ or a finite quantity;} \\ \quad (2) \text{ when } \phi_2(c) = \text{a finite quantity, and } \phi_1(c) = -\infty. \\ (b) \text{ We write } \phi_2(c) = \phi_1(c) \\ \quad (1) \text{ when } \phi_2(c) = +\infty, \text{ and } \phi_1(c) = +\infty; \\ \quad (2) \text{ when } \phi_2(c) = -\infty, \text{ and } \phi_1(c) = -\infty. \end{cases}$$

The theorem to be proved may now be stated as follows:

VIII. *Theorem* VII *still holds when* (12) *is replaced by:*

(12′) *Neither* y_1 *nor* y_2 *vanishes in the interval* $a < x < b$,

provided inequalities (13) *and* (14) *are interpreted, when necessary, according to the convention* (17).

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* We state here, for the sake of reference, the form which this theorem takes when we interchange y and z and also the subscripts 1 and 2.

VII_1. *If the conditions* (9) *are fulfilled then the following three conditions:*

Neither z_1 *nor* z_2 *vanishes in* (I),

$$\frac{y_2(a)}{z_2(a)} \leqq \frac{y_1(a)}{z_1(a)},$$

$$\frac{y_2(b)}{z_2(b)} \geqq \frac{y_1(b)}{z_1(b)},$$

cannot all be satisfied except in the special case in which $y_1/z_1 \equiv y_2/z_2$, *a case which can occur only when:*

$Q_1 \equiv Q_2$, *and* $Q_1 = Q_2 = 0$ *and* $y_1 = y_2 = 0$ *at all points where* $R_1 \neq R_2$.

The proof here is identical with that of the original theorem unless at least one of the conditions:

$$(18) \qquad \omega_2(a) = \omega_1(a) = \pm \infty,$$

$$(19) \qquad \omega_2(b) = \omega_1(b) = \pm \infty,$$

is fulfilled. These cases require further consideration.

Suppose first that (18), which is merely a special case of (13), holds but (19) does not; and let conditions (12′) and (14) also be fulfilled. Let c be any point between a and b, and apply theorem VII to the interval $c \leqq x \leqq b$. We thus see that:

$$(20) \qquad \omega_2(c) \leqq \omega_1(c).$$

Now take c so near to a that neither z_1 nor z_2 vanishes in the interval $a \leqq x \leqq c$, and to this interval apply theorem VII_1 (cf. footnote, p. 202). Since:

$$\frac{y_2(a)}{z_2(a)} = \frac{y_1(a)}{z_1(a)} = 0,$$

we see that we must have:

$$\frac{y_2(c)}{z_2(c)} \leqq \frac{y_1(c)}{z_1(c)};$$

and, therefore, since by (18) the two sides of this inequality have the same sign:

$$(21) \qquad \omega_2(c) \geqq \omega_1(c).$$

By comparing (20) and (21) we see that $\omega_2(c) = \omega_1(c)$; and this is possible, as we see by again considering the interval $c \leqq x \leqq b$, only if at every point of this interval:

$$(22) \qquad \omega_2 = \omega_1 \quad \text{and} \quad \Delta = 0.$$

Moreover, since c may be taken as near to a as we please, it is clear that (22) holds at every point of the interval $a < x \leqq b$.

In precisely the same way we see that if (19) holds but (18) does not the relations (12′) and (13) can hold only when conditions (22) are fulfilled at every point of the interval $a \leqq x < b$.

Finally if (18) and (19) both hold, and if (12′) is fulfilled, let c be any point between a and b. Then by applying what has just been proved first to the interval $a \leqq x \leqq c$, then to the interval $c \leqq x \leqq b$, it follows that $\omega_2(c) = \omega_1(c)$. But this is possible, as was just seen, only if (22) is fulfilled at all points of the interval $a < x < b$.

Thus we see in all cases that, if (12′), (13) and (14) are fulfilled, (22) must hold at all points of the interval $a < x < b$. From this the truth of theorem VIII follows at once.

By imposing on Q_1 and Q_2 the additional restriction that they both have characteristic signs, and that these signs be the same, two fundamental theorems of comparison can be deduced from theorem VIII. Let us assume for distinctness that Q_2 and therefore also Q_1 has the negative characteristic sign. In this case we see, by applying Lemma I (§ 1) to the first equation (1_i) that at a point where y_i vanishes it increases with x if z_i is positive, and decreases if z_i is negative. Since in this case (cf. theorem II) y_i has only a finite number of roots, it follows that ω_1 and ω_2 are continuous throughout (I) except at a finite number of points, and as x increases through one of these points the function in question jumps from $-\infty$ to $+\infty$.

Let us now apply theorem VIII to the interval $x_1 \leqq x \leqq x_2$ between two successive roots x_1 and x_2 of y_2. When we confine our attention to this interval we have:

$$\omega_2(x_1) = +\infty, \quad \omega_2(x_2) = -\infty.$$

Conditions (13), (14), and the first half of (12′) are therefore fulfilled, and we get the theorem:

IX. *If the conditions* (9) *are fulfilled and* Q_2 *has a negative characteristic sign, then between two successive roots of* y_2 *lies at least one root of* y_1 *unless between these roots* z_1/y_1 *and* z_2/y_2 *are identically equal, a case which can occur only when between these roots conditions* (15) *and* (16) *are fulfilled.*

This theorem is only a special case, though a particularly important one, of the following:

X. First Theorem of Comparison: *If conditions* (9) *are fulfilled and* Q_2 *has a negative characteristic sign, and:*

$$(23)\quad \begin{array}{ll} \textit{either} & y_2(a) = 0, \\ \textit{or} & y_1(a) \neq 0, \ y_2(a) \neq 0, \ \dfrac{z_2(a)}{y_2(a)} \geqq \dfrac{z_1(a)}{y_1(a)}, \end{array}$$

and if y_2 *has* n *roots* $x_1, \cdots, x_n (a < x_1 < x_2 < \cdots < x_n \leqq b)$, *then* y_1 *has at least* n *roots in the interval* $a < x \leqq x_n$, *and except when* z_1/y_1 *and* z_2/y_2 *are identically equal throughout this interval (a case which can occur only when conditions* (15) *and* (16) *are fulfilled throughout this interval) the nth root of* y_1 *to the right of* a *is less than* x_n.

That y_1 has at least n roots in the interval $a < x \leqq x_n$ follows from the fact that according to theorem IX it has at least one root in each of the intervals:

$$x_{i-1} < x \gneqq x_i \qquad (i = 2, 3, \cdots, n),$$

together with the fact that it has at least one root in the interval $a < x \leqq x_1$, as is seen by applying theorem VIII to this interval.

Suppose now that z_1/y_1 and z_2/y_2 are not identically equal throughout the interval $a < x < x_1$. Then theorem VIII shows that y_1 has at least one root in this interval, and theorem IX shows that it also has at least one root in each of the intervals:

$$x_{i-1} \leqq x < x_i \qquad (i = 2, 3, \cdots, n).$$

In this case therefore the nth root of y_1 is less than x_n.

If on the other hand $z_1/y_1 \equiv z_2/y_2$ throughout the interval $a < x < x_1$ but not throughout the interval $a < x < x_n$, then there is at least one interval between two successive roots of y_2 in which this identity does not hold. Suppose that $x_{k-1} < x < x_k$ is such an interval. Then y_1 has at least one root in this interval and also in each of the intervals:

$$a < x \leqq x_1,$$

$$x_{i-1} < x \leqq x_i \qquad (i = 2, 3, \cdots, k-1),$$

$$x_{i-1} \leqq x < x_i \qquad (i = k+1, \cdots, n).$$

Here again the nth root of y_1 is less than x_n.

That the identity $z_1/y_1 \equiv z_2/y_2$ can hold throughout the interval $a < x < x_n$ only when conditions (15) and (16) are fulfilled throughout this interval is seen by considering formulæ (10) and (11).

We come now to the

XI. Second Theorem of Comparison: *If conditions* (9) *and* (23) *are fulfilled and Q_2 has a negative characteristic sign, and if y_1 and y_2 have the same number of roots in the interval $a < x < b$ and neither of these functions vanishes at b, then:*

$$\frac{z_2(b)}{y_2(b)} > \frac{z_1(b)}{y_1(b)}, \tag{24}$$

except when $z_2/y_2 \equiv z_1/y_1$, a case which can occur only when conditions (15) *and* (16) *are fulfilled at every point of* (I).

If neither y_1 nor y_2 vanish in the interval $a < x < b$ the truth of this theorem follows at once from VIII. Otherwise let $x_1, \cdots, x_n$ be the roots of y_2 arranged in order of magnitude, and let $\bar{x}_n$ be the nth root of y_1. The first theorem of comparison tells us that $\bar{x}_n \leqq x_n$. Accordingly neither y_1 nor y_2 vanishes in the interval $x_n < x < b$, and an application of theorem VIII to this last interval shows that (24) holds unless $z_1/y_1 \equiv z_2/y_2$ throughout the last mentioned interval. This case, however, can occur only when $\bar{x}_n = x_n$; and this in turn is possible only when $z_1/y_1 \equiv z_2/y_2$ throughout the interval $a < x < x_n$.

§ 4. *Generalization by change of variable.*

Let us now introduce into equations (1) in place of y, z the new dependent variables:

$$\Phi = \phi_1 y - \phi_2 z, \tag{25}$$
$$\Psi = \psi_1 y - \psi_2 z,$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are continuous real functions of x with continuous first derivatives. We further assume that the function $\phi_1 \psi_2 - \phi_2 \psi_1$ does not vanish at any point of (I).

By differentiating equations (25) and then eliminating y, y', z, z' between the equations thus obtained and equations (25) and (1) Φ and Ψ are found to satisfy equations of the form:

$$\Phi' = \bar{P}\Phi - \bar{Q}\Psi, \tag{26}$$
$$\Psi' = \bar{R}\Phi - \bar{S}\Psi.$$

If we introduce the notation:

$$\{\alpha, \beta, \gamma, \delta\} = \alpha'\beta - \gamma\delta' - Q\alpha\gamma + P\alpha\beta + S\gamma\delta - R\beta\delta,$$

and for still greater brevity:

$$\{\alpha, \beta\} = \{\alpha, \beta, \alpha, \beta\},$$

the coefficients in (26) may be written:

$$\bar{P} = \frac{\{\phi_1, \psi_2, \psi_1, \phi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}, \qquad \bar{Q} = \frac{\{\phi_1, \phi_2\}}{\phi_1\psi_2 - \phi_2\psi_1},$$

$$\bar{R} = \frac{\{\psi_1, \psi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}, \qquad \bar{S} = \frac{\{\psi_1, \phi_2, \phi_1, \psi_2\}}{\phi_1\psi_2 - \phi_2\psi_1}.$$

Since it has been assumed that $\phi_1\psi_2 - \phi_2\psi_1$ vanishes nowhere in (I), it follows that the coefficients of (26) are continuous throughout (I). Moreover it is clear that Φ and Ψ vanish identically when, and only when, y and z do so. By applying to (26) the third theorem stated in § 1 we see that if $\Phi \equiv 0$ either $\bar{Q} \equiv 0$ or $\Psi \equiv 0$, i. e., either $\{\phi_1, \phi_2\} \equiv 0$ or $y \equiv z \equiv 0$. This is a result which concerns the function Φ only, but its proof depended on the existence of two functions ψ_1 and ψ_2 such that $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish. By letting

$$\psi_1 = \phi_2, \qquad \psi_2 = -\phi_1 \tag{27}$$

we obtain two such functions provided ϕ_1 and ϕ_2 do not vanish simultaneously. Hence the theorem:

XII. *If* ϕ_1 *and* ϕ_2 *do not vanish together, and if* $\Phi \equiv 0$, *then either* $\{\phi_1, \phi_2\} \equiv 0$ *or* $y \equiv z \equiv 0$.

By applying to equations (26) instead of to (1) the theorems of §§ 2, 3 a series of theorems is obtained the more important of which we now proceed to state.

Determining Ψ as above by equations (27) we deduce at once the following theorem from II:

XIII. *If y and z are not identically zero, and if ϕ_1 and ϕ_2 do not vanish at the same point, then if $\{\phi_1, \phi_2\}$ has a characteristic sign, Φ does not vanish an infinite number of times in (I) and when it vanishes it changes sign.*

From III and IV follows the theorem:

XIV. *If y and z are not identically zero, if $\phi_1\psi_2 - \phi_2\psi_1$ does not vanish in (I), and if $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ both have characteristic signs, then*

(a) if these signs are alike the roots of Φ and Ψ separate each other;

(b) if these signs are different neither Φ nor Ψ vanishes more than once in (I), and if one vanishes the other does not.

Let us now consider the functions:

$$\Phi_1 = \phi_1 y_1 - \phi_2 z_1, \quad \Psi_1 = \psi_1 y_1 - \psi_2 z_1,$$
$$\Phi_2 = \phi_1 y_2 - \phi_2 z_2, \quad \Psi_2 = \psi_1 y_2 - \psi_2 z_2,$$

where (y_1, z_1) and (y_2, z_2) are two solutions of (1). It is evident that a necessary and sufficient condition that (Φ_1, Ψ_1) and (Φ_2, Ψ_2) be linearly independent is that (y_1, z_1) and (y_2, z_2) be linearly independent. If then Ψ is determined by means of (27) we obtain from theorem V the result:

XV. *If (y_1, z_1) and (y_2, z_2) are two linearly independent solutions of* (1), *if ϕ_1 and ϕ_2 do not both vanish at any point of (I), and if $\{\phi_1, \phi_2\}$ has a characteristic sign, then Φ_1 and Φ_2 have no roots in common, and the roots of these two functions separate each other.*

If we now consider besides the two functions Φ and Ψ a third function:

$$X = \chi_1 y - \chi_2 z,$$

where χ_1, χ_2 are continuous real functions of x with continuous first derivatives, we may state the following theorem:

XVI. *If y and z are not identically zero; if none of the three functions:*

$$\phi_1\psi_2 - \phi_2\psi_1, \quad \psi_1\chi_2 - \psi_2\chi_1, \quad \chi_1\phi_2 - \chi_2\phi_1, \tag{28}$$

vanish in (I); and if the three functions:

$$\{\phi_1, \phi_2\}, \quad \{\psi_1, \psi_2\}, \quad \{\chi_1, \chi_2\}, \tag{29}$$

have the same characteristic sign, then the roots of the three functions Φ, Ψ, X follow each other cyclically in the order named or in the reverse order according as the product of the six functions (28) *and* (29) *has a negative or positive characteristic sign.*

As the proof of this theorem follows precisely the lines of the proof of a similar theorem which I have given in an earlier paper,* I omit it here.

The results of § 3 might also be generalized by means of the transformation (25), but for the sake of brevity we omit the statement of the theorems which might be obtained at once in this way.

§ 5. *Small variations in differential equations and initial conditions.*

We consider in this section the two systems (1_1) and (1_2) and we begin by proving that if their coefficients, which we assume to be continuous functions of x, and also the initial conditions imposed on their solutions, differ only slightly then the solutions themselves differ only slightly; that is, in more precise language:

XVII. *Two positive constants M and ϵ being given, a positive δ exists such that if:*

$$(30)\qquad \begin{gathered} |P_i| < M, \quad |Q_i| < M, \quad |R_i| < M, \quad |S_i| < M \qquad (i=1,2), \\ |P_2 - P_1| < \delta, \quad |Q_2 - Q_1| < \delta, \quad |R_2 - R_1| < \delta, \quad |S_2 - S_1| < \delta, \end{gathered}$$

and if c is any point of (I) and (y_1, z_1) and (y_2, z_2) are two solutions of (1_1) and (1_2) respectively which satisfy the conditions:

$$\begin{gathered} |y_i(c)| < M, \quad |z_i(c)| < M \qquad (i=1,2), \\ |y_2(c) - y_1(c)| < \delta, \quad |z_2(c) - z_1(c)| < \delta, \end{gathered}$$

then throughout (I):

$$|y_2 - y_1| < \epsilon, \quad |z_2 - z_1| < \epsilon.$$

To prove this theorem we use the method of successive approximations which tells us † that:

$$(31)\qquad \begin{aligned} y_i &= u_0^{(i)} + u_1^{(i)} + u_2^{(i)} + \cdots \\ z_i &= v_0^{(i)} + v_1^{(i)} + v_2^{(i)} + \cdots \end{aligned} \qquad (i=1,2),$$

where:

$$u_0^{(i)} = y_i(c), \qquad v_0^{(i)} = z_i(c),$$

$$u_j^{(i)} = \int_c^x (P_i u_{j-1}^{(i)} - Q_i v_{j-1}^{(i)})\,dx, \qquad v_j^{(i)} = \int_c^x (R_i u_{j-1}^{(i)} - S_i v_{j-1}^{(i)})\,dx,$$
$$(i=1,2;\ j=1,2,\cdots).$$

From these formulæ it follows that the nth term of each of the series (31) is numerically less than or equal to the nth term of the series

$$\sum_{j=0}^{j=\infty} M\frac{(2Ml)^j}{j!},$$

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*Cf. these Transactions, vol. 2 (1901), pp. 432–433.

† Cf. PEANO, Mathematische Annalen, vol. 32 (1888), p. 450.

where l is the length $b - a$ of the interval (I). Let us now take n so that the remainder of this last written series after the nth term is less than $\epsilon/3$. The same will be true of the absolute values of the remainders after the nth term of the four series (31). Accordingly we have:

$$|y_2 - y_1| < \sum_{j=0}^{j=n-1} |u_j^{(2)} - u_j^{(1)}| + \frac{2\epsilon}{3},$$

$$|z_2 - z_1| < \sum_{j=0}^{j=n-1} |v_j^{(2)} - v_j^{(1)}| + \frac{2\epsilon}{3}.$$

The theorem will therefore be proved if δ, which is as yet wholly unrestricted, can be so chosen that:

$$(32) \qquad |u_j^{(2)} - u_j^{(1)}| < \frac{\epsilon}{3n}, \quad |v_j^{(2)} - v_j^{(1)}| < \frac{\epsilon}{3n} \quad (j = 0, 1, \cdots, n-1).$$

Let us assume that, η being given at pleasure, there exists a δ such that:

$$(33) \qquad |u_j^{(2)} - u_j^{(1)}| < \eta, \quad |v_j^{(2)} - v_j^{(1)}| < \eta \quad (j = 0, 1, \cdots, m < n-1).$$

When $m = 0$ such a δ exists; in fact in this case we need merely take $\delta < \eta$. We shall therefore have established formulæ (33) for all values of $m < n$, and therefore also formulæ (32), by the method of mathematical induction, if we can show that by still further decreasing δ (if necessary) formulæ (33) can be made to hold when $j = m + 1$. For this purpose let δ be taken so that formulæ (33) hold when η is replaced by $\eta/4Ml$, and also so that:

$$\delta < \frac{\eta \cdot m!}{2(2Ml)^{m+1}}.$$

Now we have:

$$u_{m+1}^{(2)} - u_{m+1}^{(1)} = \int_c^x [P_2(u_m^{(2)} - u_m^{(1)}) + u_m^{(1)}(P_2 - P_1) - Q_2(v_m^{(2)} - v_m^{(1)}) - v_m^{(1)}(Q_2 - Q_1)]\,dx,$$

$$v_{m+1}^{(2)} - v_{m+1}^{(1)} = \int_c^x [R_2(u_m^{(2)} - u_m^{(1)}) + u_m^{(1)}(R_2 - R_1) - S_2(v_m^{(2)} - v_m^{(1)}) - v_m^{(1)}(S_2 - S_1)]\,dx.$$

Since:

$$|u_m^{(i)}| \leqq M\frac{(2Ml)^m}{m!}, \quad |v_m^{(i)}| \leqq M\frac{(2Ml)^m}{m!},$$

it follows from the above equations that:

$$|u_{m+1}^{(2)} - u_{m+1}^{(1)}| < \int_c^x \frac{\eta}{l}\,|dx| \leqq \eta, \quad |v_{m+1}^{(2)} - v_{m+1}^{(1)}| < \int_c^x \frac{\eta}{l}\,|dx| \leqq \eta.$$

Thus the theorem is proved.

We come next to the theorem:

XVIII. *If c is a point of (I) and* (y_1, z_1) *is a solution of* (1_1), *and in a portion* $a_1 \leqq x \leqq b_1$ *of (I)* y_1 *has just n roots* x_i:

$$a_1 < x_1 < x_2 < \cdots < x_n < b_1,$$

and if Q_1 *does not change sign in (I), then for every positive* ϵ *a positive* δ *exists, such that if* P_2, Q_2, R_2, S_2 *are continuous functions of x, satisfying the inequalities:*

$$|P_2 - P_1| < \delta, \quad |Q_2 - Q_1| < \delta, \quad |R_2 - R_1| < \delta, \quad |S_2 - S_1| < \delta,$$

and if Q_2 *has a characteristic sign,* and* (y_2, z_2) *is a solution of* (1_2) *which satisfies the conditions:*

$$|y_2(c) - y_1(c)| < \delta, \quad |z_2(c) - z_1(c)| < \delta,$$

then y_2 *has in the interval* $a_1 \leqq x \leqq b_1$ *exactly n roots* ξ_i:

$$a_1 < \xi_1 < \xi_2 < \cdots < \xi_n < b_1,$$

and these roots satisfy the inequalities:

$$|x_i - \xi_i| < \epsilon \qquad (i = 1, 2, \cdots, n).$$

In order to prove this theorem let us take a positive quantity ϵ_1 satisfying the following conditions:

$$\epsilon_1 \leqq \epsilon, \quad \epsilon_1 < x_1 - a_1, \quad \epsilon_1 < b_1 - x_n,$$

$$2\epsilon_1 < x_{i+1} - x_i \qquad (i = 1, 2, \cdots, n-1),$$

and such that z_1 does not vanish in any of the intervals:

$$(J_i) \qquad |x - x_i| \leqq \epsilon_1 \qquad (i = 1, 2, \cdots, n).$$

Another set of intervals (K_i) shall be defined by the formulæ:

$$(K_0) \qquad a_1 \leqq x \leqq x_1 - \epsilon_1,$$

$$(K_i) \qquad x_i + \epsilon_1 \leqq x \leqq x_{i+1} - \epsilon_1 \qquad (i = 1, 2, \cdots, n-1),$$

$$(K_n) \qquad x_n + \epsilon_1 \leqq x \leqq b_1.$$

Thus in each interval (J) lies one and only one root of y_1; in each interval (K) no such root. If it can be shown that the same is true of the roots of y_2 when δ is properly chosen our theorem will be proved. Let us then note the smallest value of $|z_1|$ in the intervals (J) and the smallest value of $|y_1|$ in the intervals (K), and, taking the smaller of these two quantities as the ϵ of theorem XVII,

* It is assumed here for the sake of simplicity that Q_2 has a characteristic sign throughout (I) and Q_1 does not change sign. It is, however, clearly sufficient if throughout the neighborhood of each point x_i the function Q_2 has a characteristic sign, and Q_1 does not change sign.

determine δ by means of that theorem. Then y_2 does not vanish in any interval (K) nor z_2 in any interval (J). We see then by theorem I that y_2 cannot vanish more than once in any interval (J). In order to see that it does vanish once in each interval (J) we notice that in each interval (K) y_1 and y_2 have the same sign; and therefore, since y_1 changes sign each time it vanishes (cf. theorem I), y_2 has opposite signs in two successive intervals (K) and therefore vanishes in the intervening interval (J).

A modification of the theorem just proved which is often useful is the following:

XIX. *If* (y_1, z_1) *is a solution of* (1_1), *and if in the interval* (I) y_1 *vanishes at* a *and at just* n *other points* x_i:

$$a < x_1 < x_2 < \cdots < x_n < b,$$

and if Q_1 *does not change sign in* (I), *then for every positive* ϵ *a positive* δ *exists such that if* P_2, Q_2, R_2, S_2 *are continuous functions satisfying the inequalities:*

$$|P_2 - P_1| < \delta, \quad |Q_2 - Q_1| < \delta, \quad |R_2 - R_1| < \delta, \quad |S_2 - S_1| < \delta,$$

and if Q_2 *has a characteristic sign,* and* (y_2, z_2) *is a solution of* (1_2) *which satisfies the conditions:*

$$|y_2(a)| < \delta, \quad |z_2(a) - z_1(a)| < \delta, \quad y_2(a) \cdot z_1(a) \geqq 0,$$

then y_2 *has in the interval* $a < x \leqq b$ *exactly* n *roots* ξ_i:

$$a < \xi_1 < \xi_2 < \cdots < \xi_n < b,$$

and these roots satisfy the conditions:

$$|x_i - \xi_i| < \epsilon \qquad (i = 1, 2, \cdots, n).$$

To prove this theorem we notice first that z_1 does not vanish at a as otherwise y_1 and z_1 would vanish identically. Accordingly there exists a constant a_1 satisfying the condition $a < a_1 < x_1$ and such that z_1 does not vanish at any point of the interval $a \leqq x \leqq a_1$. If then δ can be so chosen that y_2 has no root in the interval $a < x \leqq a_1$ the truth of our theorem follows at once from theorem XVIII if we let $b_1 = b$.

Let us then, in order to complete the proof, give to δ a value such that z_2 has throughout the interval $a \leqq x \leqq a_1$ the same sign as z_1. That such a choice of δ is possible follows at once from theorem XVII if in that theorem we take for ϵ a quantity less than the least value of $|z_1|$ in the interval $a \leqq x \leqq a_1$. It is now at once obvious that if $y_2(a) = 0$ the function y_2 does not vanish again in the interval $a \leqq x \leqq a_1$ as otherwise y_2 would vanish twice in an interval in which z_2 does not vanish, and this contradicts theorem I. It remains to show that, if $y_2(a) \neq 0$, y_2 does not vanish in the interval $a \leqq x \leqq a_1$. If y_2 vanishes in

*Cf. the last footnote.

this interval let x_0 be its smallest root. At this point, as an application of the first lemma of § 1 to the first equation (1_2) shows, y_2 passes from negative to positive or from positive to negative according as $z_2(x_0)$ is positive or negative. In either case z_2 and y_2 have opposite signs for values of x a little less than x_0, while at the point a z_1 and y_2, and therefore also z_2 and y_2, have the same sign. This, however, is impossible since neither y_2 nor z_2 vanishes between a and x_0. Thus our theorem is proved.

§ 6. *Differential equations involving parameters.*

The coefficients P, Q, R, S of the equations (1) shall be assumed in the present section to depend not merely on x but also on one or more parameters, $\lambda_1, \lambda_2, \cdots, \lambda_k$. These parameters we suppose confined to certain intervals finite or infinite. We assume that P, Q, R, S are continuous functions of the $k+1$ independent variables $(x;\ \lambda_1, \lambda_2, \cdots, \lambda_k)$ for all the values of these variables with which we are concerned. The solutions (y, z) of (1) will then themselves be functions of the parameters λ_i as well as of x. This will be indicated by the notation:

$$y(x;\ \lambda_1, \lambda_2, \cdots, \lambda_k), \quad z(x;\ \lambda_1, \lambda_2, \cdots, \lambda_k).$$

The following result follows at once from the theorems of the last section:*

XX. *If* (y, z) *is a solution of* (1) *such that:*

$$y(a;\ \lambda_1, \cdots, \lambda_k), \quad z(a;\ \lambda_1, \cdots, \lambda_k)$$

are continuous functions of $(\lambda_1, \cdots, \lambda_k)$ *for all values of these variables with which we are concerned, then:*

(*a*) y *and* z *are continuous functions of* $(x;\ \lambda_1, \cdots, \lambda_k)$ *for all the values of these variables with which we are concerned;*

(*b*) *if* y *is not identically zero for any set of the* λ_i*'s, and if for all the values of the parameters* λ_i *with which we are concerned* Q *has a characteristic sign, and if for all such values there exist at least* n *values of* x *in the interval* $a < x < b$ *for which* y *is zero, and if these values (or, if there are more than* n *of them, the* n *smallest ones) arranged in order of increasing magnitude are denoted by* $x_1, x_2, \cdots, x_n$*,—then* x_i $(i = 1, 2, \cdots, n)$ *is a continuous function of the independent variables* $(\lambda_1, \lambda_2, \cdots, \lambda_k)$*, provided that there exists a*

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* We state this theorem, for the sake of simplicity, only for the case in which the point c considered in the last section coincides with the point a. The portion (a) of the theorem is well known. It is stated here only for the sake of completeness. The portion (b) establishes under certain conditions the continuity of an implicit function. By restricting the range of values of the λ's it might easily have been stated in a form in which it establishes the existence of this implicit function. We are, however, considering a case which goes beyond that commonly considered, for on the one hand the function y need have no derivatives with regard to the λ's, and on the other hand $\partial y/\partial x$ may vanish at a point x_i.

constant m such that for all values for the λ's for which $y(a;\lambda_1,\cdots,\lambda_k)\neq 0$ the following inequality holds:

$$\frac{z(a;\lambda_1,\cdots,\lambda_k)}{y(a;\lambda_1,\cdots,\lambda_k)}>m. \tag{34}$$

The portion (a) of this theorem follows at once from theorem XVII; while theorem XVIII establishes the continuity of x_i for any set of values $(\lambda_1,\cdots,\lambda_k)$ for which $y(a;\lambda_1,\cdots,\lambda_k)\neq 0$.

Let us then, in order to complete the proof, consider a set of values $(\bar\lambda_1,\cdots,\bar\lambda_k)$ such that $y(a;\bar\lambda_1,\cdots,\bar\lambda_k)=0$. If we can show that a positive δ exists such that when:

$$|\lambda_i-\bar\lambda_i|<\delta \qquad (i=1,2,\cdots,k),$$

then:

$$y(a;\lambda_1,\cdots,\lambda_k)\cdot z(a;\lambda_1,\cdots,\lambda_k)\geqq 0,$$

the continuity of x_i at the point $(\bar\lambda_1,\cdots,\bar\lambda_k)$ will follow at once from XIX when we remember that for a sufficiently small δ the functions:

$$z(a;\lambda_1,\cdots,\lambda_k),\qquad z(a;\bar\lambda_1,\cdots,\bar\lambda_k),$$

have the same sign.

That a δ of the sort here required does exist may be shown as follows. Suppose no such δ exists. Then there exists an infinite sequence of points:

$$(\lambda_1^{[j]},\lambda_2^{[j]},\cdots,\lambda_k^{[j]}) \qquad (j=1,2,\cdots),$$

such that:

$$1^\circ \qquad \lim_{j=\infty}\lambda_i^{[j]}=\bar\lambda_i \qquad (i=1,2,\cdots,k),$$

$$2^\circ \quad y(a;\lambda_1^{[j]},\cdots,\lambda_k^{[j]})\cdot z(a;\lambda_1^{[j]},\cdots,\lambda_k^{[j]})<0 \qquad (j=1,2,\cdots).$$

From this it follows that:

$$\lim_{j=\infty}\frac{z(a;\lambda_1^{[j]},\cdots,\lambda_k^{[j]})}{y(a;\lambda_1^{[j]},\cdots,\lambda_k^{[j]})}=-\infty,$$

and this is in contradiction with (34).

The theorems just proved in conjunction with the Theorems of Comparison (§ 3) form a sufficient basis from which to develop with ease a theory of the system (1) when the coefficients P, Q, R, S are continuous functions of (x,λ)* of such a nature that $P+S$ is independent of λ, Q and R continually increase (or at least do not decrease) as λ increases, and, for all values of λ with which we are concerned, Q has a negative characteristic sign. We should then

* I confine my attention here, for the sake of simplicity, to the case where there is only one parameter λ. Cf., however, *Encyclopädie d. math. Wissenschaften*, vol. 2, part 1, p. 450.

consider a solution (y, z) of (1) determined by initial conditions at a in such a way that $y(a, \lambda)$ and $z(a, \lambda)$ are continuous functions of λ which do not vanish simultaneously, and such that if $y(a, \lambda)$ vanishes for one value of λ it vanishes for all larger values of λ with which we are concerned, and that when $y(a, \lambda) \neq 0$ the function $z(a, \lambda)/y(a, \lambda)$ either increases or remains constant as λ increases. Under these circumstances it follows at once from theorem X that (except in certain special cases which correspond to (15) and (16))* the roots of $y(x, \lambda)$ are continuous functions of λ which continually increase as λ increases; while from XI it follows that (except in the special cases just referred to) when $y(b, \lambda) \neq 0$ the function $z(b, \lambda)/y(b, \lambda)$ continually increases.

Further developments, for instance the proof of Theorems of Oscillation, will then follow without difficulty. Although the theorems thus obtained would be distinctly more general than STURM's theorems, the methods to be used are from this point on precisely those which I have used on another occasion † to establish a special case of STURM's results, and I shall therefore not take up the space which would be necessary to give them in full.

§ 7. *The homogeneous linear equation of the second order.*

By applying the theorems of §§ 2, 4 to the system of equations (3) a series of propositions concerning the equation of the second order (2) may be immediately deduced. The theorems thus obtained are for the most part identical with the propositions which are in part explicitly, in part implicitly contained in the third section of my paper: *On certain pairs of transcendental functions whose roots separate each other.*‡ I therefore state here merely the proposition which follows from theorem XV, this being a result which was not mentioned in the paper just referred to. It is assumed here and in what follows that p and q are throughout (I) continuous real functions of x.

If y_1 and y_2 are two linearly independent solutions of (2), *and if ϕ_1 and ϕ_2 are throughout (I) continuous real functions of x with continuous first derivatives which do not vanish together and such that the function:*

$$\phi_1'\phi_2 - \phi_1\phi_2' + \phi_1^2 + p\phi_1\phi_2 + q\phi_2^2,$$

has a characteristic sign, then the functions :

$$\phi_1 y_1 - \phi_2 y_1', \quad \phi_1 y_2 - \phi_2 y_2',$$

have no root in common, and the roots of these two functions separate each other.

* It is for instance sufficient, though not necessary, that as λ increases R does not remain constant at all points between a and the root of $y(x, \lambda)$ in question.

† Bulletin of the American Mathematical Society, April and May, 1898.

‡ These Transactions, vol. 2 (1901), p. 428. References are here given to similar but less general results of STURM.

It is, however, possible to pass from equation (2) to a system of equations of the form (1) not merely by the substitution $y' = z$ but also by an infinite number of other substitutions,* of which by far the most important is the following.

If we write with STURM:

$$K = e^{\int p\,dx}, \quad G = Kq,$$

equation (2) takes on the form:

$$\frac{d(Ky')}{dx} + Gy = 0. \tag{2'}$$

This equation is equivalent to the system of the form (1):

$$y' = \frac{1}{K} z, \quad z' = -Gy. \tag{3'}$$

By applying the theorems of §§ 2, 4 to equations (3′) instead of to equations (3) we get in general propositions equivalent to those referred to at the beginning of this section though expressed in a somewhat different form.† In the case of theorem VI, however, the result obtained is more general, viz.:

If G (or q) has a positive characteristic sign, then (except at the points, necessarily finite in number, where $y = 0$) the function Ky'/y decreases continually as x increases.‡

By applying the theorems of § 3 to equations (3′) we obtain STURM's two fundamental Theorems of Comparison.§

Finally the theorems of §§ 5, 6 may be applied with ease to the system (3) or (3′).

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

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* In general we might let:

$$\bar{y} = \phi_1 y - \phi_2 y', \quad \bar{z} = \psi_1 y - \psi_2 y';$$

and we should get for $\bar{y}$, $\bar{z}$ a system of the form (1). By using these formulæ directly in § 2 the transformation of § 4 would become unnecessary.

† It is in this form that all of STURM's results are stated.

‡ Cf. STURM, l. c., p. 159.

§ Cf. STURM, l. c. A compact statement of these theorems will be found at the end of my paper: *Application of a method of d'Alembert to a proof of Sturm's theorems of comparison*, these Transactions, vol. 1 (1900), p. 414.